

THE PROBABILITY OF RECTANGULAR UNIMODULAR MATRICES OVER $\mathbb{F}_q[x]$

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ABSTRACT. In this note, we compute the probability that a $k \times n$ matrix can be extended to an $n \times n$ invertible matrix over $\mathbb{F}_q[x]$, which turns out to be $(1 - q^{k-n})(1 - q^{k-1-n}) \cdots (1 - q^{1-n})$. Connections with Dirichlet's density theorem on the co-prime integers and its various generalizations are also presented.

1. INTRODUCTION AND MAIN RESULTS

For any set A of positive integers, its **natural density** is defined as

$$(1) \quad D(A) := \lim_{N \rightarrow \infty} \frac{|A \cap \{1, 2, \dots, N\}|}{N},$$

provided the limit exists, where $|\cdot|$ denotes the cardinality of the corresponding set. Dirichlet [2] discovered an interesting density theorem that asserts the probability of two integers to be co-prime is $6/\pi^2$, that is,

$$(2) \quad \lim_{N \rightarrow \infty} \frac{|\{(m, n) \in \mathbb{N}^2 : 1 \leq m, n \leq N, \gcd(m, n) = 1\}|}{N^2} = \zeta^{-1}(2) = \frac{6}{\pi^2},$$

where $\gcd(m, n)$ denotes the greatest common divisor of m and n , and $\zeta(s)$ is the Riemann's zeta function. Moreover, the probability of n integers to be co-prime is

$$(3) \quad \lim_{N \rightarrow \infty} \frac{|\{(m_1, \dots, m_n) \in \mathbb{N}^n : 1 \leq m_1, \dots, m_n \leq N, \gcd(m_1, \dots, m_n) = 1\}|}{N^n} = \zeta^{-1}(n).$$

Kubota and Sugita [5] gave a rigorous probabilistic interpretation to Dirichlet's density theorem. For the deep links between probability theory and number theory, please refer to Tenenbaum [12], Kubilius [4] and Kac [3].

Denote by $M_{k \times n}(R)$ the set of all $k \times n$ matrices over a ring R . A matrix $A \in M_{k \times n}(R)$ is called **unimodular** if it can be extended to an $n \times n$ invertible matrix. Using the concept of unimodular, we can give a re-statement of Dirichlet's density theorem and its generalization: the probability that a random $1 \times n$ integer matrix is unimodular is $\zeta^{-1}(n)$. Naturally, we can consider the matrix form of the Dirichlet's density theorem: the probability that a random $k \times n$ ($1 \leq k \leq n$) integer matrix is unimodular? The key is to define the analogous *natural density* for the sets of the integer matrices. Maze, Rosenthal and Wagner [7] studied this problem and obtained that the probability that a random $k \times n$ matrix is unimodular is $\prod_{j=n-k+1}^n \zeta^{-1}(j)$. Furthermore, [7] also gave some interesting historical remarks for Dirichlet's density theorem.

Then it is natural to consider the same question for matrices over polynomial rings of fields. As remarked in [7], the concept of natural density does not extend

2000 *Mathematics Subject Classification.* 11C99, 15B33, 60B15.

Key words and phrases. Natural density, unimodular matrices, Riemann's zeta function, q -zeta function, finite field, polynomial ring.

naturally to the ring $\mathbb{F}[x]$ for a general field \mathbb{F} . Fortunately, this can be done for the finite fields.

Let \mathbb{F}_q be a finite field consisting of q elements, and $\mathbb{F}_q[x]$ be the polynomial ring over \mathbb{F}_q . To enumerate $\mathbb{F}_q[x]$, let Σ be the set of all vectors $\alpha = (a_0, a_1, \dots)$ with $a_i \in \{0, 1, \dots, q-1\}$ and $a_i = 0$ for sufficiently large i . Then there is a one-to-one map $\chi : \Sigma \rightarrow \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ defined by $\chi(a_0, a_1, \dots) = \sum_{i=0}^{\infty} a_i q^i$. For convenience, we denote $\alpha_m = \chi^{-1}(m)$ and $m_\alpha = \chi(\alpha)$. Then for all $m \in \mathbb{Z}_+$, we set

$$f_m(x) := \sum_{i=0}^{\infty} a_i x^i, \text{ with } \alpha_m = (a_0, a_1, \dots).$$

By a rigorous probabilistic method, Sugita and Takanobu [10] determined the probability of two polynomials over \mathbb{F}_q to be co-prime *when q is a prime*, that is,

$$(4) \quad \lim_{N \rightarrow \infty} \frac{|\{(m, n) \in \{0, 1, \dots, N-1\}^2 : \gcd(f_m, f_n) = 1\}|}{N^2} = 1 - \frac{1}{q}.$$

Recently, several authors considered the question of the probability of n polynomials to be co-prime in $\mathbb{F}_q[x]$. Morrison [8] introduced a concept of *natural density*

$$(5) \quad \tilde{D}(S) := \lim_{d \rightarrow \infty} \frac{|S \cap \mathcal{F}_d|}{|\mathcal{F}_d|}, \quad S \subseteq \mathbb{F}_q[x],$$

where \mathcal{F}_d is the set of all the polynomials in $\mathbb{F}[x]$ with degree no more than d . He first treated this problem in a heuristic way. Then through an elegant as well as rigorous argument, he deduced the following formula,

$$(6) \quad \lim_{d \rightarrow \infty} \frac{|\{(f_1, \dots, f_n) \in \mathbb{F}_q^n[x] : \gcd(f_1, \dots, f_n) = 1, \deg(f_i) \leq d, 1 \leq i \leq n\}|}{|\{f \in \mathbb{F}_q[x] : \deg(f) \leq d\}|^n} = 1 - \frac{1}{q^{n-1}}.$$

Benjamin and Bennett [1] also studied this problem; their methods are much interesting using the Euclidean algorithm which is one of the oldest algorithms. Essentially, they also got the same conclusion.

Our object in this paper is to extend the main results in [7] to the polynomial ring $\mathbb{F}_q[x]$ for any prime power q , that is, the matrix form of the Morrison's theorem (6). Denote $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{F}[x] = \mathbb{F}_q[x]$ for short. Let $\mathcal{M} := M_{k \times n}(\mathbb{F}[x])$ be the set of all $k \times n$ matrices over $\mathbb{F}[x]$ and \mathcal{M}_N be the subset of \mathcal{M} consisting of all matrices with entries in $\{f_0(x), f_1(x), \dots, f_N(x)\}$. For any subset $S \subseteq \mathcal{M}$, we define the **natural density** of S in \mathcal{M} to be

$$(7) \quad D(S) := \lim_{N \rightarrow \infty} \frac{|S \cap \mathcal{M}_N|}{|\mathcal{M}_N|},$$

if the above limit (7) exists. Note that, in the special case $k = 1$, the limit in (5) is just the limit of a subsequence in (7) with $N = q^d$. Thus our definition of natural density is a more rigorous one.

To describe our result, i.e., to determine the natural density of $k \times n$ unimodular matrices over $\mathbb{F}[x]$, we introduce the following **q -zeta function**

$$(8) \quad \zeta_q(j) := \prod_f \left(1 - \frac{1}{q^{j \deg(f)}}\right)^{-1} = \prod_{m=1}^{\infty} \left(1 - \frac{1}{q^{jm}}\right)^{-\varphi_m},$$

where f goes through all irreducible polynomials in $\mathbb{F}[x]$ and φ_m is the number of irreducible polynomials in $\mathbb{F}[x]$ with degree m . In the above statements and what follows, **by irreducible polynomials we always mean monic irreducible polynomials (the constant polynomial 1 is excluded as usual)**. Now we are in the position to give our main result.

Theorem 1. *Let E be the set of all $k \times n$ unimodular matrices over $\mathbb{F}[x]$, then the natural density of E is*

$$(9) \quad D(E) = \prod_f \left(1 - \frac{1}{q^{j \deg(f)}}\right) = \prod_{m=1}^{\infty} \left(1 - \frac{1}{q^{jm}}\right)^{\varphi_m} = \prod_{j=n-k+1}^n \zeta_q^{-1}(j).$$

Remark 2. *In other words, the probability that a $k \times n$ polynomial matrix can be completed by an $(n-k) \times n$ matrix into an $n \times n$ invertible matrix over $\mathbb{F}[x]$ is $\prod_{j=n-k+1}^n \zeta_q^{-1}(j)$. And via introducing the q -zeta function $\zeta_q(s)$, we obtain a similar formula to the one in [7] for integer matrices*

$$D(\tilde{E}) = \prod_{j=n-k+1}^n \zeta^{-1}(j),$$

where \tilde{E} denotes the set of all $k \times n$ unimodular matrices over \mathbb{Z} .

Remark 3. *There is an interesting equation*

$$(10) \quad \prod_f (1 - t^{\deg(f)})^{-1} = \sum_{l=0}^{\infty} q^l t^l = \frac{1}{1-qt},$$

where f is over all irreducible polynomials. For an interpretation of this equation, see [8]. Putting $t = q^{-j}$, $j \in \mathbb{N}$ in (10), we get

$$(11) \quad \zeta_q^{-1}(j) = 1 - \frac{1}{q^{j-1}}.$$

Hence, we obtain an explicit formula for the natural density of E ,

$$(12) \quad D(E) = \left(1 - \frac{1}{q^{n-k}}\right) \left(1 - \frac{1}{q^{n-k+1}}\right) \cdots \left(1 - \frac{1}{q^{n-1}}\right),$$

and an interesting identity concerning $\{\varphi_m\}$,

$$(13) \quad \prod_{m=1}^{\infty} \left(1 - \frac{1}{q^{jm}}\right)^{\varphi_m} = 1 - \frac{1}{q^{j-1}}, \quad \forall j \in \mathbb{N}.$$

Remark 4. *In particular, if we put $k = 1$, Theorem 1 shows that the probability that n polynomials over the finite field \mathbb{F} to be co-prime is*

$$\zeta_q^{-1}(n) = \prod_{m=1}^{\infty} \left(1 - \frac{1}{q^{nm}}\right)^{\varphi_m} = 1 - \frac{1}{q^{n-1}}.$$

This extends Dirichlet's density theorem (2) and yields the same conclusion as Morrison [8] and Benjamin and Bennett [1]. However, noticing that we have defined the concept of natural density in a different way from them, as we mentioned before, one may find that our result actually is stronger than theirs.

We end this section with some remarks on the square case $k = n$. One may notice that our proof of Theorem 1 in §2 does not apply to this case (also, $\zeta_q(1)$ is not well defined in this case). However, we have the following result,

Proposition 5. *The natural density of $n \times n$ unimodular matrices over $\mathbb{F}_q[x]$ is 0.*

Remark 6. *The proof of Proposition 5 is quite simple and similar to the proof of Lemma 5 in [7], so we omit the details. If we make the convention that $\zeta_q^{-1}(1) = 0$, Theorem 1 also holds for $k = n$.*

2. PROOF OF MAIN RESULT

In this section, we shall give the proof of Theorem 1. Before this, we need some preparations.

Let \mathcal{P} be a finite set of irreducible polynomials in $\mathbb{F}[x]$ and $\widehat{\mathcal{P}}$ be the set of all irreducible polynomials in $\mathbb{F}[x]$. Denote by $E_{\mathcal{P}}$ the set of all matrices $A \in \mathcal{M} = M_{k \times n}(\mathbb{F}[x])$ such that the gcd of all full rank minors (i.e., minors of rank k) is co-prime to all elements in \mathcal{P} . Recall that E is the set of all unimodular matrices in \mathcal{M} , that is, each matrix in E can be extended to an $n \times n$ invertible matrix. It is well known that $A \in \mathcal{M}$ is unimodular if and only if the gcd of all full rank minors is 1 (See [9], [11] and [13] for more details). Thus we have $E = \bigcap_{\mathcal{P}} E_{\mathcal{P}}$.

Lemma 7. *Let $E_{\mathcal{P}}$ be defined as above, then we have*

$$(14) \quad D(E_{\mathcal{P}}) = \prod_{j=0}^{k-1} \prod_{f \in \mathcal{P}} (1 - q^{(j-n) \deg(f)}) = \prod_{j=n-k+1}^n \prod_{f \in \mathcal{P}} (1 - q^{-j \deg(f)}).$$

Proof. Denote $f_{\mathcal{P}} := \prod_{f \in \mathcal{P}} f$ and $d_{\mathcal{P}} = \deg(f_{\mathcal{P}})$. For any positive integer N , consider the maps,

$$(15) \quad \pi : \mathcal{M}_N \longrightarrow M_{k \times n}(\mathbb{F}[x]/(f_{\mathcal{P}}))$$

and

$$(16) \quad \psi : M_{k \times n}(\mathbb{F}[x]/(f_{\mathcal{P}})) \longrightarrow M_{k \times n} \left(\prod_{f \in \mathcal{P}} \mathbb{F}[x]/(f) \right) \longrightarrow \prod_{f \in \mathcal{P}} M_{k \times n}(\mathbb{F}[x]/(f)),$$

where $(f_{\mathcal{P}})$ and (f) denote the ideals generated by $f_{\mathcal{P}}$ and f respectively. Note that π is the canonical projection via modulo $f_{\mathcal{P}}$ and ψ is a composition of two canonical isomorphisms as \mathbb{F} vector spaces.

Taken any $A \in \mathcal{M}_N$. It is easy to see that $A \in E_{\mathcal{P}}$ if and only if the component of $\psi \circ \pi(A)$ in $M_{k \times n}(\mathbb{F}[x]/(f))$ is full rank for each $f \in \mathcal{P}$. Since $f \in \mathcal{P}$ is irreducible, we know that $\mathbb{F}[x]/(f) \simeq \mathbb{F}_{q^{\deg(f)}}$. Let $F_{q^{\deg(f)}}$ denote the set of all full rank $k \times n$ matrices over the finite field $\mathbb{F}_{q^{\deg(f)}}$. It is well known that $|F_{q^{\deg(f)}}| = \prod_{j=0}^{k-1} (q^{n \deg(f)} - q^{j \deg(f)})$ (See [6] page 455, for example).

First suppose that $N = mq^{d_{\mathcal{P}}} - 1$ for some $m \in \mathbb{N}$. Then it is easy to see

$$\{f_l(x) : 0 \leq l \leq N\} = \{f_s(x)x^{d_{\mathcal{P}}} + f_t(x) : 0 \leq s \leq m-1, 0 \leq t \leq q^{d_{\mathcal{P}}} - 1\}.$$

For any fixed $0 \leq s \leq m-1$, the following projection is one-to-one:

$$\{f_s(x)x^{d_{\mathcal{P}}} + f_t(x) : 0 \leq t \leq q^{d_{\mathcal{P}}} - 1\} \longrightarrow \mathbb{F}[x]/(f_{\mathcal{P}})$$

and the canonical projection

$$\{f_l(x) : 0 \leq l \leq N\} \longrightarrow \mathbb{F}[x]/(f_{\mathcal{P}})$$

is m -to-one. As a result, the projection map π in (15) is m^{kn} -to-one. Thus we obtain that

$$(17) \quad \begin{aligned} |E_{\mathcal{P}} \cap \mathcal{M}_N| &= m^{kn} \cdot \prod_{f \in \mathcal{P}} |F_{q^{\deg(f)}}| \\ &= m^{kn} \cdot \prod_{f \in \mathcal{P}} \prod_{j=0}^{k-1} q^{n \deg(f)} (1 - q^{(j-n) \deg(f)}) \\ &= (mq^{d_{\mathcal{P}}})^{kn} \cdot \prod_{f \in \mathcal{P}} \prod_{j=0}^{k-1} (1 - q^{(j-n) \deg(f)}) \end{aligned}$$

Now suppose that N is any positive integer. There exists some $m, r \in \mathbb{Z}_+$ such that $N + 1 = mq^{d_P} + r$ with $0 \leq r < q^{d_P}$. For convenience, set $\tilde{N} := mq^{d_P} - 1$. Then by the definition of the natural density, we have

$$\begin{aligned} D(E_{\mathcal{P}}) &= \lim_{N \rightarrow \infty} \frac{|E_{\mathcal{P}} \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ (18) \quad &= \lim_{N \rightarrow \infty} \frac{|E_{\mathcal{P}} \cap \mathcal{M}_{\tilde{N}}| + |E_{\mathcal{P}} \cap (\mathcal{M}_N - \mathcal{M}_{\tilde{N}})|}{(N + 1)^{kn}}. \end{aligned}$$

Note that $|\mathcal{M}_N - \mathcal{M}_{\tilde{N}}| \leq rkn(N + 1)^{kn-1}$, which gives that

$$(19) \quad \lim_{N \rightarrow \infty} \frac{|E_{\mathcal{P}} \cap (\mathcal{M}_N - \mathcal{M}_{\tilde{N}})|}{(N + 1)^{kn}} = 0.$$

Thus, from (18), (19) and (17), we obtain

$$\begin{aligned} D(E_{\mathcal{P}}) &= \lim_{N \rightarrow \infty} \frac{|E_{\mathcal{P}} \cap \mathcal{M}_{\tilde{N}}|}{(N + 1)^{kn}} \\ &= \lim_{N \rightarrow \infty} \frac{(N + 1 - r)^{kn} \prod_{f \in \mathcal{P}} \prod_{j=0}^{k-1} (1 - q^{(j-n) \deg(f)})}{(N + 1)^{kn}} \\ &= \prod_{f \in \mathcal{P}} \prod_{j=0}^{k-1} (1 - q^{-(n-j) \deg(f)}) \\ &= \prod_{j=n-k+1}^n \prod_{f \in \mathcal{P}} (1 - q^{-j \deg(f)}). \end{aligned}$$

This completes the proof of Lemma 7. \square

Proof of Theorem 1. Suppose that $k < n$. For any irreducible polynomial $f \in \mathbb{F}_q[x]$, let $H_f \subseteq \mathcal{M}$ be the set of all matrices whose gcd of all full rank minors is divisible by f . Denote $q_f := q^{\deg(f)}$, then by Lemma 7,

$$\begin{aligned} D(H_f) &= 1 - D(E_{\{f\}}) \\ &= 1 - \prod_{j=n-k+1}^n (1 - q^{-j \deg(f)}) \\ &< 1 - \left(1 - \sum_{j=n-k+1}^n q_f^{-j} \right) \\ &< \sum_{j=n-k+1}^{\infty} q_f^{-j} = \frac{1}{q_f^{n-k}(q_f - 1)} \leq \frac{2}{q_f^2}. \end{aligned}$$

Let \mathcal{P}_t be the set of all irreducible polynomials with degree no more than t and denote $E_t = E_{\mathcal{P}_t}$. Since

$$(E_t \setminus E) \subseteq \bigcup_{f \in \hat{\mathcal{P}} \setminus \mathcal{P}_t} H_f,$$

we have

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|(E_t \setminus E) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} &\leq \limsup_{N \rightarrow \infty} \frac{|(\bigcup_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} H_f) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\
&\leq \limsup_{N \rightarrow \infty} \frac{\sum_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} |H_f \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\
&\leq \sum_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} \limsup_{N \rightarrow \infty} \frac{|H_f \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\
&= \sum_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} D(H_f) \leq \sum_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} \frac{2}{q_f^2} \\
&= \sum_{f \in \widehat{\mathcal{P}} \setminus \mathcal{P}_t} \frac{2}{q^{2 \deg(f)}} = \sum_{m=t+1}^{\infty} \frac{2\varphi_m}{q^{2m}}.
\end{aligned}$$

It is well known that all irreducible polynomials with degree m can divide $x^{q^m} - x$, which has no multiple roots. Thus $m\varphi_m \leq q^m$ and

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|(E_t \setminus E) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} &\leq \sum_{m=t+1}^{\infty} \frac{2}{mq^m} \\
&\leq \sum_{m=t+1}^{\infty} \frac{2}{q^m} = \frac{2}{q^t(q-1)}.
\end{aligned}$$

Now note that $E \cap \mathcal{M}_N \subseteq E_t \cap \mathcal{M}_N$ and $E \cap \mathcal{M}_N = E_t \cap \mathcal{M}_N - (E_t \setminus E) \cap \mathcal{M}_N$, which imply that

$$\begin{aligned}
\liminf_{N \rightarrow \infty} \frac{|E \cap \mathcal{M}_N|}{|\mathcal{M}_N|} &\geq \liminf_{N \rightarrow \infty} \frac{|E_t \cap \mathcal{M}_N|}{|\mathcal{M}_N|} - \limsup_{N \rightarrow \infty} \frac{|(E_t \setminus E) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\
&\geq D(E_t) - \frac{2}{q^t(q-1)},
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{N \rightarrow \infty} \frac{|E \cap \mathcal{M}_N|}{|\mathcal{M}_N|} &\leq \limsup_{N \rightarrow \infty} \frac{|E_t \cap \mathcal{M}_N|}{|\mathcal{M}_N|} - \liminf_{N \rightarrow \infty} \frac{|(E_t \setminus E) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\
&\leq D(E_t),
\end{aligned}$$

for all $t \in \mathbb{N}$. Let t tend to ∞ , and recall that the number of monic irreducible polynomials with degree m in $\mathbb{F}[x]$ is φ_m . From Lemma 7 and the definition of q -zeta function, we can conclude that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \frac{|E \cap \mathcal{M}_N|}{|\mathcal{M}_N|} &= \lim_{t \rightarrow \infty} D(E_t) \\
&= \lim_{t \rightarrow \infty} \prod_{j=n-k+1}^n \prod_{f \in \mathcal{P}_t} (1 - q^{-j \deg(f)}) \\
&= \lim_{t \rightarrow \infty} \prod_{j=n-k+1}^n \prod_{m=1}^t (1 - q^{-jm})^{\varphi_m} \\
&= \prod_{j=n-k+1}^n \prod_{m=1}^{+\infty} (1 - q^{-jm})^{\varphi_m} = \prod_{j=n-k+1}^n \zeta_q^{-1}(j).
\end{aligned}$$

This completes the proof of Theorem 1. \square

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